SURE WINS, SEPARATING PROBABILITIES AND THE REPRESENTATION OF LINEAR FUNCTIONALS

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Abstract. We discuss conditions under which a convex cone $\mathcal{K} \subset \mathbb{R}^\Omega$ admits a finitely additive probability m such that $\sup_{k \in \mathcal{K}} m(k) \leq 0$. Based on these, we characterize those linear functionals that are representable as finitely additive expectations. A version of Riesz decomposition based on this property is obtained as well as a characterisation of positive functionals on the space of integrable functions.

1. Introduction

A long standing approach to probability, originating from the seminal work of de Finetti, views set functions P as maps which assign to each set (event) E in some class $\mathscr{A}$ the price $P(E)$ for betting 1 dollar on the occurrence of E. A set function generating a betting system which admits no sure wins was termed coherent by de Finetti who proved in [5] that a set function on a finite algebra $\mathscr{A}$ is coherent if and only if it is a probability. Since then this result has been extended and generalized by various authors, among which Heath and Sudderth [9], Lane and Sudderth [10] and Regazzini [11], to name but a few; Borkar et al. [4] is a more recent example.

In this paper we examine the absence of sure wins for a convex cone $\mathcal{K}$ of real valued functions on some arbitrary set Ω, obtaining conditions for the existence of a finitely additive probability measure m such that $\sup_{k \in \mathcal{K}} m(k) \leq 0$, i.e. a *separating probability*. The special case in which $\mathcal{K}$ is the kernel of some linear functional leads to the characterization of those functionals that admit the representation as finitely additive expectations, a topic addressed by Berti and Rigo in a highly influential paper [2]. A version of Riesz decomposition based on this representation property is obtained.

Throughout the paper Ω will be a fixed set, 2^Ω its power set, $\mathbb{R}^\Omega$ and $\mathfrak{B}$ the classes of real-valued and bounded functions on Ω respectively (the latter endowed with the topology induced by the supremum norm). All spaces of real-valued functions on Ω (e.g. bounded or integrable) will be considered as equipped with pointwise ordering, with no further mention. The lattice notation f^+ and f^- wll be used to denote the positive and negative parts of $f \in \mathbb{R}^\Omega$. The term probability is used to designate positive, finitely additive set functions m on 2^Ω (in symbols, $m \in ba_+$) such that $m(\Omega) = 1$. The symbol $\mathbb{P}_{ba}$ will be used to denote the family of all probability measures; $\mathbb{P}$ the subfamily of all countably additive probability measures. If $\mathscr{A} \subset 2^\Omega$ then by $\mathscr{S}(\mathscr{A})$ and $\mathfrak{B}(\mathscr{A})$ we denote the class of simple functions generated by $\mathscr{A}$ and its closure in $\mathfrak{B}$. We adopt the useful convention of identifying single-valued functions with their range so that, for example, we may use 1 either to denote an element of $\mathbb{R}$, or a function f on Ω such that $f(\omega) = 1$ for all

Date: October 27, 2008.

2000 *Mathematics Subject Classification.* Primary 28A25, Secondary 28A12, 28C05.

Key words and phrases. Daniell theorem, Finitely additive probability, Finitely additive supermartingales, Integral representation of linear functionals, Riesz decomposition.

$\omega \in \Omega$. In the terminology adopted throughout the following sections a *sure win* is defined to be an element of $\mathbb{R}^\Omega$ which exceeds 1.

We recall that $f \in \mathbb{R}^\Omega_+$ is integrable with respect to $m \in ba_+$, in symbols $f \in L(m)$, if and only if

$$\sup\{m(h) : h \in \mathfrak{B}, 0 \le h \le f\} < \infty \tag{1.1}$$

The integral $m(f)$ coincides then with the left hand side of (1.1); moreover, $f \wedge n$ converges to f in $L(m)$ [7, theorem III.3.6]. A special notion of convergence in $L(m)$ will be used in the following. A sequence $\langle f_n \rangle_{n\in\mathbb{N}}$ is said to converge orderly in $L(m)$ to f if $f_n \in L(m)$ for all n and there exists a pointwise decreasing sequence $\langle \bar{f}_n \rangle_{n\in\mathbb{N}}$ in $L(m)_+$ which converges to 0 in $L(m)$ and is such that $|f_n - f| \le \bar{f}_n$ for $n \ge 1$. It is easily seen that if a sequence $\langle f_n \rangle_{n\in\mathbb{N}}$ converges to f orderly in $L(m)$ then so does any of each subsequences; moreover, the space of sequences converging orderly in $L(m)$ is a vector space.

2. Separating Probabilities

Fix a convex cone $\mathcal{K} \subset \mathbb{R}^\Omega$ (that is $f+g,\ \lambda f \in \mathcal{K}$ whenever $f, g \in \mathcal{K}$ and $\lambda \ge 0$) and let $\mathcal{K}_b = \{k \in \mathcal{K} : k^- \in \mathfrak{B}\}$. For each $f \in \mathbb{R}^\Omega$ let $\mathcal{U}(f) = \{\alpha \in \mathbb{R} : \alpha + k \ge f \text{ for some } k \in \mathcal{K}\}$ and define $\pi_\mathcal{K} : \mathbb{R}^\Omega \to \overline{\mathbb{R}}$ as

$$\pi_\mathcal{K}(f) = \inf\{\alpha : \alpha \in \mathcal{U}(f)\}^1 \tag{2.1}$$

From (2.1), $\pi_\mathcal{K}$ is monotonic, $\pi_\mathcal{K}(\lambda + f) = \lambda + \pi_\mathcal{K}(f)$ for each $\lambda \in \mathbb{R}$ and $f \in \mathbb{R}^\Omega$ and $\pi_\mathcal{K}(f) \le \sup_{\omega\in\Omega} f(\omega)$ (as $0 \in \mathcal{K}$). Since $\mathcal{K}$ is a convex cone, $\mathcal{U}(f) + \mathcal{U}(g) \subset \mathcal{U}(f+g)$ and $\mathcal{U}(\lambda f) = \lambda\mathcal{U}(f)$ for $\lambda > 0$: $\pi_\mathcal{K}$ is thus subadditive and positively homogeneous; moreover, $\pi_\mathcal{K}(k) \le 0$ for all $k \in \mathcal{K}$.

Given that $\pi_\mathcal{K}(0) = 2\pi_\mathcal{K}(0) \le 0$ and $\pi_\mathcal{K}(1) = \pi_\mathcal{K}(0) + 1$, then $\pi_\mathcal{K}(0) > -\infty$ implies $\pi_\mathcal{K}(0) = 0$ and $\pi_\mathcal{K}(1) = 1$. Moreover there is $k \in \mathcal{K}$ such that $k \ge 1$ if and only if $\pi_\mathcal{K}(1) \le 0$. Thus:

Lemma 1. *Let $\mathcal{K} \subset \mathbb{R}^\Omega$ be a convex cone. Then the following are equivalent: (i) $\pi_\mathcal{K}(0) > -\infty$, (ii) $\pi_\mathcal{K}(0) = 0$, (iii) $\pi_\mathcal{K}(1) = 1$, (iv) $\mathcal{K}$ contains no sure wins.*

Denote $L(\pi_\mathcal{K}) = \left\{f \in \mathbb{R}^\Omega : \pi_\mathcal{K}(|f|) < \infty\right\}$. It is clear that $\mathfrak{B} \subset L(\pi_\mathcal{K})$. Define also

$$\mathscr{M}(\mathcal{K}) = \left\{m \in \mathbb{P}_{ba} : \mathcal{K} \subset L(m), \quad \sup_{k\in\mathcal{K}} m(k) \le 0\right\} \tag{2.2}$$

and let $\mathscr{M}(\mathcal{K}_b)$ be defined likewise. We shall refer to elements of $\mathscr{M}(\mathcal{K})$ as *separating probabilities* for $\mathcal{K}$. It is clear that if $m \in \mathscr{M}(\mathcal{K}_b)$ then $L(\pi_\mathcal{K}) \subset L(m)$.

Proposition 1. *Let $\mathcal{K} \subset \mathbb{R}^\Omega$ be a convex cone. Then $\mathscr{M}(\mathcal{K}_b)$ is non empty if and only if $\mathcal{K}$ contains no sure wins.*

Proof. Assume that $\mathcal{K}$ contains no sure wins. By Lemma 1 and the Hahn Banach Theorem, we may find a linear functional ϕ on $\mathfrak{B}$ such that $\phi \le \pi_\mathcal{K}$ on $\mathfrak{B}$ and $\phi(1) = 1$. If $f \in \mathfrak{B}_+$ then $\phi(f) = -\phi(-f) \ge -\pi_\mathcal{K}(-f) \ge 0$. Therefore ϕ is positive and, since continuous [7, V.2.7], it may be represented as the expectation with respect to some $m \in \mathbb{P}_{ba}$. If $f \in L(\pi_\mathcal{K})_+$, the left hand side of (1.1) is bounded by $\pi_\mathcal{K}(f)$ so that $L(\pi_\mathcal{K}) \subset L(m)$. Then $\mathcal{K}_b \subset L(m)$ and

$$m(k) = \lim_n m(k \wedge n) \le \pi_\mathcal{K}(k) \le 0 \qquad k \in \mathcal{K}_b$$

so that $m \in \mathscr{M}(\mathcal{K}_b)$. If $m \in \mathscr{M}(\mathcal{K}_b)$ and $k \in \mathcal{K}$ is a sure win, then $k \in \mathcal{K}_b$ and $m(k) \le 0$, a contradiction. □

[1] The functional $\pi_\mathcal{K}$ is well known in mathematical finance under the name of *superhedging price*.

A classical application of Proposition 1 considers the collection $\mathcal{K}$ of all finte sums of the form $\sum_n a_n(\mathbf{1}_{F_n} - \lambda(F_n))$ where $a_1, \ldots, a_N$ are real numbers, $F_1, \ldots, F_N$ are elements of some $\mathscr{A} \subset 2^\Omega$ and $\lambda : \mathscr{A} \to \mathbb{R}$. It is then clear that $\mathcal{K}$ admits no sure wins if and only if there is $m \in \mathbb{P}_{ba}$ such that $m|\mathscr{A} = \lambda$. If the sums in $\mathcal{K}$ are allowed to admit countably many terms provided $\sum_n |a_n \lambda(F_n)| < \infty$, then m will possess the additional property that $m(\bigcup_n F_n) = \sum_n m(F_n)$ when $\langle F_n \rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathscr{A}$. This informal statement is essentially a reformulation of [9, theorems 5 and 6, p. 2074][2]. It admits an interesting generalisation to the case of concave integrals, a special case of the monotone integral of Choquet treated, e.g., in [8].

Definition 1. *An extended real-valued functional γ on a convex cone $\mathscr{L} \subset \mathbb{R}^\Omega$ is a concave integral if it is positively homogeneous, monotone, superadditive and such that $\gamma(c + f) = \gamma(c) + \gamma(f)$ when $c, f \in \mathscr{L}$ and c is a constant.*

If γ is a concave integral on $\mathscr{L}$ we define its core to be the set

$$\Gamma(\gamma) = \{\lambda \in \ ba_+ : \ \mathscr{L} \subset L(\lambda),\ \gamma(f) \leq \lambda(f),\ f \in \mathscr{L}\} \tag{2.3}$$

The following Lemma is essentially a restatement of a result of Shapley [12, theorem 2, p. 18]. It characterises the properties of a concave integral in terms of its core.

Lemma 2. *Let $\mathscr{L} \subset \mathbb{R}^\Omega$ be a convex cone that contains the constants and is such that $f \in \mathscr{L}$ implies $f^+ \in \mathfrak{B}$. Let $\gamma : \mathscr{L} \to \mathbb{R}$ be a concave integral and $\gamma(1) > 0$. Then $\gamma(1) < \infty$ if and only if for each convex set $C \subset \mathscr{L} \cap \mathfrak{B}$ such that $\gamma(C) \equiv \sup_{f\in C} \gamma(f) < \infty$ there exists $\lambda_C \in \Gamma(\gamma)$ such that*

$$\sup_{f\in C} \lambda_C(f) = \gamma(C) \tag{2.4}$$

Proof. Assume, upon normalization, $\gamma(1) = 1$ and suppose that

$$\alpha(k - \gamma(C)) \geq 1 + \sum_{n=1}^N (f_n - \gamma(f_n)) \tag{2.5}$$

for some choice of $\alpha \geq 0$, $k \in C$ and $f_n \in \mathscr{L}$, $n = 1, \ldots, N$. The value under γ of the left hand side of (2.5) is less than 0 while that of the right hand side exceeds 1, contradicting monotonicity. Thus the collection $\mathcal{K}_C$ of finite sums of the form $\sum_{1\leq n\leq N}(\gamma(f_n) - f_n) + \alpha(k - \gamma(C))$ for α, k and f_n, $n = 1, \ldots, N$ as above contains no sure win; moreover, it is a convex cone of uniformly lower bounded functions on Ω. According to Proposition 1, there exists $\lambda_C \in \mathscr{M}(\mathcal{K}_C)$: thus, $\lambda_C(f) \geq \gamma(f)$ for each $f \in \mathscr{L}$ (i.e. $\lambda_C \in \Gamma(\gamma)$) and $\lambda_C(k) \leq \gamma(C)$ whenever $k \in C$, proving (2.4). The converse is obvious. □

Lemma 2 has an interesting implication.

Corollary 1. *Let $\mathcal{T}$ be a collection of subsets of some set T, with $\{T\} = \tau_0 \in \mathcal{T}$. For each $\tau \in \mathcal{T}$, let $\mathscr{L}_\tau$ be a linear subspace of $\mathfrak{B}$ with $1 \in \mathscr{L}_{\tau_0}$ and ϕ_τ a linear functional on $\mathscr{L}_\tau$. The following are equivalent:*

(*i*) *the collection $(\phi_\tau : \tau \in \mathcal{T})$ is coherent in the sense that*[3]

$$\sup\left\{\sum_{n=1}^N \phi_{\tau_n}(b_n) : \ b_n \in \mathscr{L}_{\tau_n},\ \sum_{n=1}^N b_n \mathbf{1}_{\tau_n} \leq 1,\ N \in \mathbb{N}\right\} < \infty$$

[2] However we do not restrict $\mathscr{A}$ nor λ. Heath and Sudderth seem to suggest that the existence of m need not exclude sure wins while it is clear that this cannot be the case. A less general version of this result was also proved, with different methods, in [4, theorem 2, p. 420]

[3] The inequality that follows is meant to hold pointwise in $\Omega \times T$

(*ii*) *there exists* $\lambda \in ba(\Omega\times T)$ *such that* $\lambda(b\mathbf{1}_\tau)=\phi_\tau(b)$ *for each* $b\in\mathscr{L}_\tau$ *and* $\tau\in\mathcal{T}$

Proof. Assume (*i*) and define the functional γ on $\mathfrak{B}(\Omega\times T)$ implicitly as

$$\gamma(b)=\sup\left\{\sum_{n=1}^N\phi_{\tau_n}(b_n):\ b_n\in\mathscr{L}_{\tau_n},\ \sum_{n=1}^N b_n\mathbf{1}_{\tau_n}\le b,\ N\in\mathbb{N}\right\} \tag{2.6}$$

It is readily seen that γ is monotone, superadditive and positively homogeneous. (*i*) implies that $\gamma(1)<\infty$ and that γ is real-valued while $1\in\mathscr{L}_{\tau_0}$ implies that γ is additive relative to the constants. (*ii*) follows from (*i*), Lemma 2 and the fact that each $\mathscr{L}_\tau$ is a linear space: simply choose $\lambda\in\Gamma(\gamma)$. If λ is as in (*ii*) and $\sum_{n=1}^N b_n\mathbf{1}_{\tau_n}\le 1$ where $b_n\in\mathscr{L}_{\tau_n}$ $n=1,\dots,N$ then $\sum_{n=1}^N\phi_{\tau_n}(b_n)=\lambda\left(\sum_{n=1}^N b_n\mathbf{1}_{\tau_n}\right)\le\|\lambda\|$. □

Remark 1. *Writing* $\tau\le\upsilon$ *when* $\tau\subset\upsilon$ *makes of course* $\mathcal{T}$ *into a partially ordered set. If* $(\phi_\tau:\tau\in\mathcal{T})$ *is coherent in the sense of Corollary 1 and if* $(\mathscr{L}_\tau:\tau\in\mathcal{T})$ *is increasing in* τ *then necessarily* $\phi_\upsilon|\mathscr{L}_\tau\ge\phi_\tau$ *whenever* $\tau,\upsilon\in\mathcal{T}$ *and* $\tau\le\upsilon$. *This conclusion has a direct application to the theory of finitely additive supermartingales, treated in* [6].

Much of this section rests on the conclusion, established in Proposition 1, that $\mathcal{K}_b$ admits a separating probability in the absence of sure wins. This result, however, does not have an extension to $\mathcal{K}$ of a corresponding simplicity. To this end we shall need some results on the representation of linear functionals, to be developed in the next section.

3. The Representation of Linear Functionals

It is the purpose of this section to establish conditions for a linear functional ϕ on some linear subspace $\mathscr{L}$ of $\mathbb{R}^\Omega$ with $1\in\mathscr{L}$ to admit the representation

$$\phi(f)=\phi(1)m(f)\qquad f\in\mathscr{L} \tag{3.1}$$

for some $m\in ba$ such that $\mathscr{L}\subset L(m)$, referred to as a *representing measure* for ϕ. We use the symbols $\mathcal{K}^\phi$ and $\mathcal{K}^\phi_b$ to denote the sets $\{f\in\mathscr{L}:\phi(f)=0\}$ and $\{f\in\mathcal{K}^\phi:f^-\in\mathfrak{B}\}$, respectively. If $\phi(1)\neq 0$, then $\mathcal{K}^\phi_b=\{f-\phi(1)^{-1}\phi(f):f\in\mathscr{L},f^-\in\mathfrak{B}\}$. Thus if $\mathscr{L}$ is a vector sublattice of $\mathbb{R}^\Omega$ then $m\in\mathscr{M}(\mathcal{K}^\phi_b)$ implies $\mathscr{L}\subset L(m)$ and $\phi(f)=\phi(1)m(f)$ for every $f\in\mathscr{L}\cap\mathfrak{B}$ (which clarifies the connection between separating probabilities and representing measures).

The content of this section, as will soon become clear, owes much to the work of Berti and Rigo [2].

Theorem 1. *Let* $\mathscr{A}\subset 2^\Omega$ *be an algebra,* $\mu\in ba(\mathscr{A})$, $\mathscr{L}$ *a vector sublattice of* $L(\mu)$ *with* $1\in\mathscr{L}$ *and* ϕ *a positive linear functional on* $\mathscr{L}$. *Denote by* $\mathscr{L}^*$ *the set of limit points of sequences from* $\mathscr{L}$ *which converge orderly in* $L(\mu)$. *The following are equivalent.*

(*i*) ϕ *extends to a monotone function* $\phi^*:\mathscr{L}^*\to\mathbb{R}$;

(*ii*) $\lim_n\phi(h_n)=0$ *whenever* $\langle h_n\rangle_{n\in\mathbb{N}}$ *is a sequence in* $\mathscr{L}$ *which converges to* 0 *orderly in* $L(\mu)$;

(*iii*) $-\infty<\lim_n\phi(g_n)\le\lim_n\phi(f_n)<\infty$ *whenever* $\langle f_n\rangle_{n\in\mathbb{N}}$ *and* $\langle g_n\rangle_{n\in\mathbb{N}}$ *are sequences in* $\mathscr{L}$ *which converge orderly in* $L(\mu)$ *to* f *and* g *respectively, with* $f\ge g$;

(*iv*) ϕ *admits a positive representing measure* m *such that* $m^*(h)\equiv\lim_n m(h_n)$ *exists in* $\mathbb{R}$ *and is unique for every sequence* $\langle h_n\rangle_{n\in\mathbb{N}}$ *in* $\mathscr{L}$ *which converges to* h *orderly in* $L(\mu)$.

Moreover, if ϕ is a positive linear functional on a vector sublattice $\mathscr{L}$ of $\mathbb{R}^\Omega$ with $1 \in \mathscr{L}$ then there exists a unique positive linear functional $\phi^\perp$ on $\mathscr{L}$ such that $\phi^\perp(1) = 0$ and that

$$\phi(f) = \phi(1)m(f) + \phi^\perp(f) \qquad f \in \mathscr{L} \tag{3.2}$$

for some $m \in ba_+$ satisfying $\mathscr{L} \subset L(m)$.

Proof. Let $\langle h_n \rangle_{n\in\mathbb{N}}$ be as in (ii) and $\langle \bar{h}_n \rangle_{n\in\mathbb{N}}$ be a decreasing sequence in $L(m)$ converging to 0 in $L(m)$ and such that $\bar{h}_n \geq |h_n|$, $n = 1, 2, \ldots$. Fix a sequence $\langle \alpha_n \rangle_{n\in\mathbb{N}}$ in $\mathbb{R}_+$ such that $\lim_n \alpha_n = \infty$. Any subsequence of $\langle h_n \rangle_{n\in\mathbb{N}}$ admits a further subsequence (still denoted by $\langle h_n \rangle_{n\in\mathbb{N}}$ for convenience) such that $\sum_n \alpha_n \|h_n\| < \infty$. Fix $\eta > 0$ arbitrarily and set

$$h_n^\eta = (h_n - \eta)^+, \; g_k^\eta = \sum_{n\leq k} \alpha_n h_n^\eta \quad \text{and} \quad g^\eta = \sum_n \alpha_n h_n^\eta \tag{3.3}$$

Then, $\left\{\sum_{n>k} \alpha_n h_n^\eta > \epsilon\right\} \subset \{\bar{h}_k \geq \eta\}$ and $\left\|\sum_{k<n\leq k+p} \alpha_n h_n^\eta\right\| \leq \sum_{n>k} \alpha_n \|h_n^\eta\| \leq \sum_{n>k} \alpha_n \|h_n\|$. Thus, $\langle g_k^\eta \rangle_{k\in\mathbb{N}}$ is an increasing sequence in $\mathscr{L}$ which converges orderly in $L(\mu)$ to $g^\eta \in \mathscr{L}^*$ [7, theorem III.3.6]. If (i) holds then $\alpha_n^{-1}\phi^*(g^\eta) \geq \phi(h_n^\eta) \geq \phi(h_n) - \eta\phi(1)$ so that $\lim_n \phi(h_n) = 0$, i.e. (ii) holds as well. Let $\langle g_n \rangle_{n\in\mathbb{N}}$ and $\langle f_n \rangle_{n\in\mathbb{N}}$ be as in (iii). The inequality $f_n - g_n \geq (f_n - f) + (g - g_n)$ together with (ii) induces the conclusion that $(f_n - g_n)^-$ converges to 0 orderly in $L(\mu)$ and thus that $\liminf_n \phi(f_n) = \liminf_n\{\phi(g_n) + \phi((f_n - g_n)^+)\} \geq \liminf_n \phi(g_n)$. The case in which $\langle g_n \rangle_{n\in\mathbb{N}}$ is a subsequence of $\langle f_n \rangle_{n\in\mathbb{N}}$ suggests that $\liminf_n \phi(f_n) = \limsup_n \phi(f_n)$. If $\lim_n \phi(f_n) = \infty$ then one may select a subsequence $\langle f_{n_k} \rangle_{k\in\mathbb{N}}$ such that, letting $h_k = f_{n_{k+1}} - f_{n_k}$, $\lim_k \phi(h_k) = \infty$. However this contrasts with (ii) since the sequence $\langle h_k \rangle_{k\in\mathbb{N}}$ converges to 0 orderly in $L(\mu)$. This proves (iii). In the general case in which $\mathscr{L}$ is a vector sublattice of $\mathbb{R}^\Omega$, fix $f \in \mathscr{L}_+$ and choose $m \in \mathscr{M}(\mathcal{K}_b^\phi)$ if $\phi(1) > 0$, or $m = 0$ otherwise. Then,

$$\phi(f) = \lim_n \phi(f \wedge n) + \lim_n \phi(f - (f \wedge n)) = \phi(1)m(f) + \phi^\perp(f) \tag{3.4}$$

a conclusion which extends to general $f \in \mathscr{L}$ by considering f^+ and f^- separately. The functional $\phi^\perp$, as defined implicitly in (3.4), is clearly positive, linear and such that $\phi^\perp(1) = 0$. Decomposition (3.2) thus exists. If $\phi(f) = \phi(1)v(f) + \psi^\perp(f)$ were another decomposition such as (3.2), with $v \in ba_+$, $\mathscr{L} \subset L(v)$ and $\psi^\perp$ a positive, linear functional on $\mathscr{L}$ with $\psi^\perp(1) = 0$, then $f \in \mathscr{L}_+$ would imply

$$(\phi^\perp - \psi^\perp)(f) = \lim_n (\phi^\perp - \psi^\perp)(f - (f \wedge n)) = \phi(1) \lim_n (m + v)(f - (f \wedge n)) = 0$$

which proves uniqueness of (3.2). Returning to the case $\mathscr{L} \subset L(\mu)$, if (iii) holds, then it is obvious from (3.4) that $\phi^\perp = 0$; in addition the limit $\lim_n m(h_n)$ exists in $\mathbb{R}$ for each sequence $\langle h_n \rangle_{n\in\mathbb{N}}$ in $\mathscr{L}$ which converges orderly in $L(\mu)$ and does not depend but on the limit point h. □

One noteworthy implication of Theorem 1, obtained by replacing $\mathscr{L}$ with $L(\mu)$, is the following

Theorem 2. *Let $\mathscr{A} \subset 2^\Omega$ be an algebra and $\mu \in ba(\mathscr{A})$. Every positive linear functional ϕ on $L(\mu)$ admits a positive representing measure m such that $\lim_n m(h_n) = 0$ for every sequence $\langle h_n \rangle_{n\in\mathbb{N}}$ in $L(\mu)$ which converges to 0 orderly in $L(\mu)$.*

Given that $L(\mu)$ is a normed Riesz space, its dual space is a vector lattice [1, theorem 12.1, p. 175]. Thus Theorem 2 also implies that continuous linear functionals, decomposing as the difference of two positive linear functionals, admit a representing measure [2, theorem 7, p. 3255].

Another application concerns more general functionals. In fact it is clear that the implication $(i)\rightarrow(ii)$ in Theorem 2 does not require ϕ to be linear.

Theorem 3. *Let $\mathscr{L} \subset \mathbb{R}^\Omega$ be either (i) a Banach lattice containing the constants or (ii) $\mathscr{L} = L(\mu)$ for some $\mu \in ba(\mathscr{A})$ and some algebra $\mathscr{A} \subset 2^\Omega$. Assume that $\phi : \mathscr{L} \to \mathbb{R}$ is a monotone functional such that*

$$\lim_n \inf_{\{f \in \mathscr{L}: \phi(f) > \eta\}} \phi(nf) = \infty \qquad \eta > 0 \tag{3.5}$$

and, under (ii),

$$\lim_{k \downarrow 0} \sup_{f \in \mathscr{L}} \{\phi(f) - \phi(f - k)\} = 0 \tag{3.6}$$

Then, $\limsup_n \phi(h_n) \le 0$ when $\langle h_n \rangle_{n \in \mathbb{N}}$ converges to 0 in norm or, under (ii), orderly in $L(\mu)$. In particular, convex, monotone functionals on a Banach lattice are continuous.

Proof. Each subsequence of $\langle h_n \rangle_{n \in \mathbb{N}}$ contains a further subsequence for which it is possible to define g_k^η and g^η as in (3.3). Under (i), $\langle g^\eta \rangle_{k \in \mathbb{N}}$ converges to g^η in norm for all $\eta \ge 0$; under (ii) only for $\eta > 0$. In either case we conclude that $\phi(g^\eta) \ge \phi(\alpha_n h_n^\eta) \ge \phi(\alpha_n(h_n - \eta))$ and, given (3.5), $\liminf_n \phi(h_n - \eta) \le 0$. Choosing $\eta = 0$ under (i) or exploiting (3.6) under (ii) and recalling that the intial choice of the subsequence was arbitrary, we conclude that $\limsup_n \phi(h_n) \le 0$. It is clear that a convex functional ϕ meets (3.5), (3.6) and, by monotonicity, $|\phi(h) - \phi(h_n)| \le \phi(|h_n - h|)$. □

Given the preceding results, it is now easy to extend Proposition 1 to $\mathcal{K}$.

Corollary 2. *Let $\mathcal{K} \subset \mathbb{R}^\Omega$ be a convex cone. Then $\mathscr{M}(\mathcal{K})$ is non empty if and only if there exist an algebra $\mathscr{A} \subset 2^\Omega$ and $\mu \in \mathbb{P}_{ba}(\mathscr{A})$ such that $\mathcal{K} \subset L(\mu)$ and that the closure $\overline{C}^\mu$ of $C = \mathcal{K} - \mathscr{S}(\mathscr{A})_+$ in the norm topology of $L(\mu)$ admits no sure wins.*

Proof. If $\mu \in \mathscr{M}(\mathcal{K})$ then μ is a separating measure for $\overline{C}^\mu$ which rules out sure wins. As for sufficiency, observe that ordinary separation theorems imply the existence of a continuous linear functional $\phi : L(\mu) \to \mathbb{R}$ such that $\sup_{f \in \overline{C}^\mu} \phi(f) \le 0$ and $1 = \phi(1)$. Given that $\mathcal{K}$ contains the origin, $-\mathscr{S}(\mathscr{A})_+ \subset C$ so that ϕ is positive on $\mathscr{S}(\mathscr{A})$ and, since $\mathscr{S}(\mathscr{A})_+$ is dense in $L(\mu)_+$ and ϕ is $L(\mu)$ continuous, it is positive over the whole of $L(\mu)$. The claim follows from Theorem 2. □

Corollary 2 is related to a result of Yan [13], where $\mathcal{K} \subset L(P)$ and P is countably additive.

The representation (3.1) extends beyond $L(\mu)$.

Corollary 3. *Let $\mathscr{L} \subset \mathbb{R}^\Omega$ be a linear space. A linear functional ϕ on $\mathscr{L}$ admits a representing measure if and only if there exists $\mu \in ba$ such that $\mathscr{L} \subset L(\mu)$ and ϕ is continuous with respect to the norm topology of $L(\mu)$. If, in addition, ϕ is positive and $\mathscr{L}$ a vector sublattice of $\mathbb{R}^\Omega$, there exists a positive representing measure.*

Proof. The direct implcation is obvious. For the converse, let $\mu \in ba$ be as in the statement and denote by $\bar{\phi}$ the continuous, linear extension of ϕ to $L(\mu)$. If $\mathscr{L}$ is a vector lattice and ϕ is positive, the inequality $\phi(f) \le \bar{\phi}(f^+)$ implies that such extension may be chosen to be positive and continuous. In either case the claim follows from Theorem 2. □

Daniell theorem also follows easily.

Corollary 4. *Let $\mathscr{L}$ be a vector sublattice of $\mathbb{R}^\Omega$ containing 1 and ϕ a positive linear functional on $\mathscr{L}$. Then $\lim_n \phi(f_n) = 0$ for every sequence $\langle f_n \rangle_{n \in \mathbb{N}}$ in $\mathscr{L}$ which decreases to 0 pointwise if and only if ϕ admits a representing measure m which is countably additive in restriction to the σ algebra generated by $\mathscr{L}$.*

Proof. Consider the case $\phi \neq 0$, the claim being otherwise trivial. Then, by (3.2), $\phi(1) > 0$ and ϕ admits a representing probability m. Let $\mathscr{A} = \left\{ E \subset \Omega : \inf_{\{g \in \mathscr{L} : g \geq \mathbf{1}_E\}} m(g) = \sup_{\{f \in \mathscr{L} : f \leq \mathbf{1}_E\}} m(f) \right\}$ and consider a decreasing sequence $\langle E_n \rangle_{n \in \mathbb{N}}$ in $\mathscr{A}$ with $\bigcap_n E_n = \varnothing$. For each $\eta > 0$ there are sequences $\langle f_n \rangle_{n \in \mathbb{N}}$ and $\langle g_n \rangle_{n \in \mathbb{N}}$ in $\mathscr{L}_+$ with $g_n \geq \mathbf{1}_{E_n} \geq f_n$ and $m(f_n) \geq m(g_n) - \eta 2^{-n}$. Let $h_n = \inf_{\{k \leq n\}} f_k$. $m(h_1) \geq m(g_1) - \eta 2^{-1}$; if $m(h_{n-1}) \geq m(g_{n-1}) - \eta \sum_{k=1}^{n-1} 2^{-k}$ for some n then, $h_{n-1} + f_n = h_n + (h_{n-1} \vee f_n) \leq h_n + g_{n-1}$ implies

$$m(h_n) \geq m(f_n) + m(h_{n-1}) - m(g_{n-1}) \geq m(f_n) - \eta \sum_{k=1}^{n-1} 2^{-k} \geq m(g_n) - \eta \sum_{k=1}^{n} 2^{-k}$$

Thus the sequence $\langle f_n \rangle_{n \in \mathbb{N}}$ may be chosen to be decreasing to 0 and such that $m(f_n) \geq m(g_n) - \eta$ for each n. Then, $0 = \lim_n m(f_n) \geq \lim_n m(E_n) - \eta$. It is well known that $\mathscr{A}$ is an algebra and that $\mathscr{L} \cap \mathfrak{B} \subset \mathfrak{B}(\mathscr{A})$, see e.g. [3, p. 774]. Thus, $m|\mathscr{A}$ admits a countably additive extension to $\sigma\mathscr{A}$ and this, in turn, an extension μ to 2^Ω. Since μ and m coincide on $\mathscr{A}$, μ is another representing measure for ϕ. The converse is a straightforward implication of monotone convergence. $\square$

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